MATH 162A Review: Vector Calculus

Facts to Know:

The chain rule: Let $u = f(x_1, \cdots, x_n)$ and let each $x_j = x_j(t_1, \cdots, t_m)$. Then

$$\boxed{\frac{\partial u}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_k} \cdot \frac{\partial x_k}{\partial t_j}.} = \sum_{i=0}^{n} \frac{\partial u}{\partial x_k} \boxed{(x_1, \cdots x_n)} \frac{\partial x_k}{\partial t_j}$$

Under the Einstein convention, it can be written as

$$u_{t_j} = u_{x_k}(x_k)_{t_j} = u_k(x_k)_j.$$

$$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_k} \cdot \frac{\partial x_k}{\partial t_j}$$

The product rule on vector-valued functions.

$$f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

1.
$$\frac{\partial(f \cdot g)}{\partial t} = \frac{\partial f}{\partial t} \cdot g + f \cdot \frac{\partial g}{\partial t} \quad \frac{\partial g}{\partial t} \cdot f$$

$$\frac{\partial f}{\partial t} =$$

2.
$$\frac{\partial(f \times g)}{\partial t} = \frac{\partial f}{\partial t} \times g + \boxed{f \times \frac{\partial g}{\partial t}}$$

3.
$$\frac{\partial (AB)}{\partial t} = \frac{\partial A}{\partial t} \cdot B + A \cdot \frac{\partial B}{\partial t}$$

$$\frac{\partial f}{\partial u}$$

Examples:

$$\alpha'(t) \cdot \alpha(t) = 0$$

1. Prove that if $\|\alpha(t)\|^2 = 2$, then $\langle \alpha'(t), \alpha(t) \rangle = 0$.

$$\alpha(t) \cdot \alpha(t) = 2$$

$$\frac{\partial}{\partial t}(\alpha(t) \cdot \alpha(t)) = 0$$

$$\frac{\partial}{\partial t}(\alpha(t) \cdot \alpha(t)) = \frac{\partial \alpha}{\partial t} \cdot \alpha + \alpha \cdot \frac{\partial \alpha}{\partial t}$$

$$= 2 \frac{\partial \alpha}{\partial t} \cdot \alpha$$

$$\alpha'(t) \cdot \alpha = 0$$

2. Prove that
$$\frac{d(e^{At})}{dt} = Ae^{At} = e^{At}A.$$

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$
$$= I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots$$

$$\frac{d}{dt}(e^{At}) = A + tA^2 + \frac{1}{2!}t^2A^3 + \cdots$$
$$= A(I + tA + \frac{1}{2!}t^2A^2 + \cdots) = Ae^{At}$$

$$\boxed{\frac{d}{dt}(e^{A(t)}) \neq A'(t)e^{A(t)} \neq e^{A(t)}A'(t)}$$

$$e^{A(t)} = I + A(t) + \frac{1}{2}A(t)^2 + \frac{1}{6}A(t)^3 + \cdots$$

$$\frac{d}{dt}A(t)^3 = 3A(t)^2A'(t)$$

$$\frac{d}{dt}A(t)^3 = A'(t)A^2(t) + A(t)A'(t)A(t) + A(t)^2A'(t)$$

$x' = Ax$ A is a matrix. $x = e^{At}\cdot c$

$x' = A(t)\cdot x$ we don't have a formula for the solution